

LIMITS OF SOME WEIGHTED CESARO AVERAGES

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ABSTRACT. We investigate the existence of the limit of some high order weighted Cesaro averages.

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1. INTRODUCTION

Motivated by potential applications to several branches of the mathematics, we study the possible convergence of high order weighted Cesaro means of the type

$$(1.1) \quad \frac{1}{n^p} \sum_{k=1}^n b_k f(k/n),$$

where $p > 0$ and $f : (0, 1] \rightarrow \mathbb{R}$, provided $(b_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ is a p -mean convergent sequence:

$$\lim_n \frac{1}{n^p} \sum_{k=1}^n b_k = b \in \mathbb{C}.$$

Averages like those in (1.1) naturally appear in Ergodic Theory. They also play a role in Probability, for example in the investigation of the central limit (see e.g. [10]), as well as in Infinite Dimensional Analysis in managing the so-called Lévy Laplacian (cf. [12]) and exotic, i.e. high order ones, see e.g. [3] and the references cited therein. Cesaro averages as above might find natural applications also in Harmonic Analysis, Linear Algebra and Matrix Theory, Numerical Analysis, Number Theory and in other sectors of pure and applied mathematics.

The convergence of the mean in (1.1) depends on the conditions imposed on the function f , which are listed in our main result in Section 2. For example, we get convergence for the simple cases

$$f(x) = x^q, \quad f(x) = (1 - x)^q, \quad q > 0,$$

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which leads to the results in Section 3 concerning averages of multi-indices sequences.

The weighted averages of multi-indices sequences appear in managing some quantum central limit theorems, when the sequence of mean covariances is not constant but at least convergent, and an order structure on some indices affects the value of the so-called mixed moments. Indeed, Propositions 3.1 and 3.2, which quite surprisingly lead to results which cannot be reflected, may be naturally exploited in Anti-Monotone and Monotone cases (see e.g [6, 7, 13]). In order to get a flavour of the several kinds of mixed moments naturally emerging in Quantum Probability and the associated problem of their computation, the reader is referred to [1, 2, 5, 8] and the references cited therein.

The last section is devoted to counterexamples which explain that all the conditions imposed on our results are essentially optimal.

We end by noticing that particular cases of averages considered here appear in Section 5 of [4] (see also [3]), where also several continuous versions of averages are investigated.

2. LIMITS OF WEIGHTED CESARO MEANS

In the present note we suppose that the set of natural numbers does not contain 0:

$$\mathbb{N} := \{1, 2, \dots, n, \dots\}.$$

We start with some elementary notations by denoting for each function $f : (0, 1] \rightarrow \mathbb{R}$, a sequence $\mathbf{b} := (b_n)_{n \in \mathbb{N}} \subset \mathbb{C}$, and finally $p \in (0, +\infty)$,

$$M_{\mathbf{b}, f; p}(n) := \frac{1}{n^p} \sum_{k=1}^n b_k f(k/n)$$

some useful high order weighted Cesaro means. For any sequence \mathbf{b} , by $|\mathbf{b}|$ we denote the sequence $(|b_n|)_{n \in \mathbb{N}}$. A sequence \mathbf{b} is said to be *p-mean convergent* if the sequence $(M_{\mathbf{b}, 1; p}(n))_{n \in \mathbb{N}}$ of its Cesaro *p*-averages is convergent, where 1 stands for the constant function $f = 1$ identically. When $p = 1$, we recover the usual setting concerning the arithmetic means. It is easy to show that, if \mathbf{b} is *p-mean convergent* then $b_n = o(n^p)$ for $n \rightarrow +\infty$.

Let $f : (0, 1] \rightarrow \mathbb{R}$ be a monotone function. Define on $(0, 1]$ the possible infinite Borel measure $|df|$ induced by the Stieltjes integral with respect to f if it is increasing, of by $-f$ if f is decreasing, see e.g. [14], Section 12.3.

The following result is useful in the sequel:

Lemma 2.1. *Let \mathbf{a} and \mathbf{b} be convergent and p -mean convergent sequences with $\lim_n a_n = a$ and $\lim_n M_{\mathbf{b},1;p}(n) = b$, respectively. Suppose that*

$$(2.1) \quad M_{|\mathbf{b}|,1;p}(n) \leq B, \quad n \in \mathbb{N},$$

then the product sequence \mathbf{ab} is p -mean convergent with

$$\lim_n M_{\mathbf{ab},1;p}(n) = ab.$$

Proof. Fix $\varepsilon > 0$ and choose l_0 such that $n > l_0$ implies $|a_n - a| < \varepsilon$. We get for $n > l_0$,

$$\begin{aligned} \left| M_{\mathbf{ab},1;p}(n) - ab \right| &\leq \left(\frac{l_0}{n} \right)^p \left(\left| M_{\mathbf{ab},1;p}(l_0) \right| + \left| a M_{\mathbf{b},1;p}(l_0) \right| \right) \\ &\quad + \left| a \left(M_{\mathbf{b},1;p}(n) - b \right) \right| + \varepsilon B. \end{aligned}$$

We then have

$$\limsup_n \left| M_{\mathbf{ab},1;p}(n) - ab \right| \leq \varepsilon B,$$

which leads to the assertion being ε arbitrary. \square

Here, there is our main result:

Theorem 2.2. *Fix a p -mean convergent sequence \mathbf{b} with $\lim_n M_{\mathbf{b},1;p}(n) = b$, and a monotone function $f : (0, 1] \rightarrow \mathbb{R}$ such that $f \in L^1((0, 1], x^{p-1}dx)$ and $x^p \in L^1((0, 1], |df|)$. Then*

$$\lim_n M_{\mathbf{b},f;p}(n) = bp \int_0^1 x^{p-1} f(x) dx.$$

Proof. We can suppose, without loosing generality, that f is decreasing by passing possibly to the opposite function, and positive by possibly adding a constant. Under the last hypotheses, for each $\varepsilon > 0$ there exists n_0 such that, if $n > n_0$

$$0 \leq \sum_{k=1}^{\lfloor \frac{n}{n_0} \rfloor} f\left(\frac{k}{n}\right) \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p \right] \leq p \int_0^{1/n_0} x^{p-1} f(x) dx \leq \varepsilon,$$

where $[x]$ is the unique integer such that $[x] \leq x < [x] + 1$ for any arbitrary real x . We then argue that

$$\begin{aligned}
0 &\leq p \int_0^1 x^{p-1} f(x) dx - \sum_{k=1}^n f\left(\frac{k}{n}\right) \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p \right] \\
&\leq p \int_0^{1/n_0} x^{p-1} f(x) dx + \sum_{k=1}^{[\frac{n}{n_0}]} f\left(\frac{k}{n}\right) \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p \right] \\
&\quad + \left\{ p \int_{1/n_0}^1 x^{p-1} f(x) dx - \sum_{k=[\frac{n}{n_0}]+1}^n f\left(\frac{k}{n}\right) \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p \right] \right\} \\
&\leq 2\varepsilon + \left\{ p \int_{1/n_0}^1 x^{p-1} f(x) dx - \sum_{k=[\frac{n}{n_0}]+1}^n f\left(\frac{k}{n}\right) \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p \right] \right\} \\
&\rightarrow 2\varepsilon
\end{aligned}$$

for $n \rightarrow +\infty$, since one recognises the last term as the Riemann-Stieltjes sum of

$$\int_0^1 f(x) dx^p = p \int_0^1 f(x) x^{p-1} dx.$$

As $\varepsilon > 0$ is arbitrary, we conclude that

$$(2.2) \quad \lim_n \sum_{k=1}^n f\left(\frac{k}{n}\right) \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p \right] = p \int_0^1 x^{p-1} f(x) dx.$$

With

$$c_n := M_{\mathbf{b},1;p}(n) - b, \quad n \in \mathbb{N},$$

we get

$$\begin{aligned}
M_{\mathbf{b},f;p}(n) &= c_n f(1) + b \sum_{k=1}^n f\left(\frac{k}{n}\right) \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p \right] \\
&\quad + \sum_{k=2}^n c_{k-1} \left(\frac{k-1}{n}\right)^p \left[f\left(\frac{k-1}{n}\right) - f\left(\frac{k}{n}\right) \right].
\end{aligned}$$

For each $\varepsilon > 0$, let n_0 such that $n > n_0$ implies $|c_n| < \varepsilon$. Then for every n sufficiently big,

$$\begin{aligned} & \left| \sum_{k=2}^n c_{k-1} \left(\frac{k-1}{n} \right)^p \left[f\left(\frac{k-1}{n} \right) - f\left(\frac{k}{n} \right) \right] \right| \\ & \leq \sum_{k=2}^{n_0+1} |c_{k-1}| \left(\frac{k-1}{n} \right)^p \left[f\left(\frac{k-1}{n} \right) - f\left(\frac{k}{n} \right) \right] \\ & \quad + \sum_{k=n_0+2}^n |c_{k-1}| \left(\frac{k-1}{n} \right)^p \left[f\left(\frac{k-1}{n} \right) - f\left(\frac{k}{n} \right) \right] \\ & < \sup_n |c_n| \int_0^{\frac{n_0+1}{n}} x^p |df(x)| + \varepsilon \int_0^1 x^p |df(x)|, \end{aligned}$$

which goes to 0 as $n \rightarrow +\infty$, because $\varepsilon > 0$ is arbitrary. Collecting the last computation with (2.2), we get the result. \square

3. SOME MULTI-DIMENSIONAL CASES

The present section is devoted to the investigation of some ergodic limits of multi-dimensional Cesaro averages which may appear in the study of Quantum Central Limit Theorems as those considered in [7].

Proposition 3.1. *Let \mathbf{b} be a p -mean convergent sequence satisfying (2.1) with $\lim_n M_{\mathbf{b},1;p}(n) = b$, and $(a_{k_1,\dots,k_m})_{k_1,\dots,k_m \in \mathbb{N}} \subset \mathbb{C}$ a multi-indices sequence such that for $q > 0$,*

$$\lim_n \frac{1}{n^q} \sum_{1 \leq k_1, \dots, k_m \leq n} a_{k_1, \dots, k_m} = a.$$

Then

$$\lim_n \frac{1}{n^{p+q}} \sum_{k=1}^n b_k \sum_{1 \leq k_1, \dots, k_m \leq k} a_{k_1, \dots, k_m} = \frac{abp}{p+q}.$$

Proof. Notice that

$$\frac{1}{n^{p+q}} \sum_{k=1}^n b_k \sum_{1 \leq k_1, \dots, k_m \leq k} a_{k_1, \dots, k_m} = M_{\mathbf{a}\mathbf{b}, x^q; p}(n),$$

where

$$a_k := \frac{1}{k^q} \sum_{1 \leq k_1, \dots, k_m \leq k} a_{k_1, \dots, k_m}, \quad k \in \mathbb{N},$$

defines the sequence \mathbf{a} which is supposed to be convergent. The proof now follows from Lemma 2.1 and Theorem 2.2. \square

Recall that the Euler's Beta and Gamma functions are defined respectively as

$$\underline{(z, t)} := \int_0^1 x^{z-1} (1-x)^{t-1} dx, \quad \operatorname{Re}(z), \operatorname{Re}(t) > 0,$$

$$\Gamma(z) := \int_0^{+\infty} x^{z-1} e^{-x} dx, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

Such special functions are related by the celebrated identity

$$(3.1) \quad \underline{(z, t)} = \frac{\Gamma(z)\Gamma(t)}{\Gamma(z+t)},$$

see e.g. [9].

The functions above appear in the following result concerning the tail-average.

Proposition 3.2. *Let $(a_{k_1, \dots, k_m})_{k_1, \dots, k_m \in \mathbb{N}}$ and \mathbf{b} be a multi-indices sequence and a sequence respectively, satisfying all the hypotheses of Proposition 3.1. If in addition,*

$$(3.2) \quad a_{k_1-h, \dots, k_m-h} = a_{k_1, \dots, k_m}$$

for any $k_1, \dots, k_m \in \mathbb{N}$ and $h < \min\{k_1, \dots, k_m\}$, then

$$(3.3) \quad \lim_n \frac{1}{n^{p+q}} \sum_{k=1}^n b_k \sum_{k+1 \leq k_1, \dots, k_m \leq n} a_{k_1, \dots, k_m} = ab \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)}.$$

Proof. Notice that (3.2) gives

$$\sum_{k+1 \leq k_1, \dots, k_m \leq n} a_{k_1, \dots, k_m} = \sum_{1 \leq k_1, \dots, k_m \leq n-k} a_{k_1, \dots, k_m}$$

and, consequently,

$$\begin{aligned} & \frac{1}{n^{p+q}} \sum_{k=1}^n b_k \sum_{k+1 \leq k_1, \dots, k_m \leq n} a_{k_1, \dots, k_m} \\ &= \frac{1}{n^{p+q}} \sum_{k=1}^n b_k (n-k)^q \left[\frac{1}{(n-k)^q} \sum_{1 \leq k_1, \dots, k_m \leq n-k} a_{k_1, \dots, k_m} - a \right] \\ &+ \frac{a}{n^{p+q}} \sum_{k=1}^n b_k (n-k)^q. \end{aligned}$$

From Proposition 2.2, one has

$$\begin{aligned} \lim_n \frac{1}{n^{p+q}} \sum_{k=1}^n b_k (n-k)^q &= bp \int_0^1 x^{p-1} (1-x)^q dx \\ &= b \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)}, \end{aligned}$$

the last equality coming from (3.1) and $\Gamma(z+1) = z\Gamma(z)$.

The thesis then follows once one shows

$$\frac{1}{n^{p+q}} \sum_{k=1}^n b_k (n-k)^q \left[\frac{1}{(n-k)^q} \sum_{1 \leq k_1, \dots, k_m \leq n-k} a_{k_1, \dots, k_m} - a \right]$$

is infinitesimal for $n \rightarrow \infty$. Indeed, since for any $\varepsilon > 0$, there is $l_0 \in \mathbb{N}$ such that for any $h \geq l_0$

$$\left| \frac{1}{h^q} \sum_{1 \leq k_1, \dots, k_m \leq h} a_{k_1, \dots, k_m} - a \right| \leq \varepsilon,$$

one has for each $k = 1, \dots, n - l_0$,

$$\left| \frac{1}{(n-k)^q} \sum_{1 \leq k_1, \dots, k_m \leq n-k} a_{k_1, \dots, k_m} - a \right| \leq \varepsilon.$$

Thus, denoting by $M > 0$ a uniform bound for the sequence of the multiple of Cesaro means of (a_{k_1, \dots, k_m}) , by (2.1) one finds

$$\begin{aligned} & \left| \frac{1}{n^{p+q}} \sum_{k=1}^n b_k (n-k)^q \left[\frac{1}{(n-k)^q} \sum_{1 \leq k_1, \dots, k_m \leq n-k} a_{k_1, \dots, k_m} - a \right] \right| \\ & \leq \frac{\varepsilon}{n^p} \sum_{k=1}^{n-l_0} |b_k| \left(\frac{n-k}{n} \right)^q \\ & + \left| \frac{1}{n^p} \sum_{k=n-l_0+1}^n b_k \left(\frac{n-k}{n} \right)^q \left[\frac{1}{(n-k)^q} \sum_{1 \leq k_1, \dots, k_m \leq n-k} a_{k_1, \dots, k_m} - a \right] \right| \\ & \leq \left[\varepsilon + 2M \left(\frac{l_0}{n} \right)^q \right] B. \end{aligned}$$

The proof is achieved as ε is arbitrary. \square

4. SOME COUNTEREXAMPLES

We end the present note by showing some counterexamples concerning the average-convergence of sequences.

We start by noticing that in Theorem 2.2, the case with \mathbf{b} identically equal to 1 and $p = 1$ corresponds simply to ask whether the sequence

of the Riemann sums of a L^1 -function f , made partitioning the interval $[0, 1]$ in n subintervals of uniform length $1/n$ and Riemann integrable on all the subintervals $[\varepsilon, 1]$, converges to the integral of f . The following simple counterexample (which can be easily modified to achieve the continuous case)

$$f = \sum_{n=1}^{+\infty} n^2 \chi_{\{1/n\}}$$

tells us that it is not always the case, even if one imposes mild natural conditions on f .

Now we pass to see that the convergence of $\frac{1}{n} \sum_{k=1}^n |b_k|$ does not imply that \mathbf{b} is mean-convergent. Let \mathbf{b} be the sequence defined as

$$\mathbf{b} := \overbrace{1}^{2^0}, \overbrace{-1, -1}^{2^1}, \overbrace{1, 1, 1, 1}^{2^2}, \overbrace{-1, \dots, -1}^{2^3}, \dots$$

Define, for each integer n ,

$$m_n := 2 \cdot 4^n - 1, \quad h_n := 4^{n+1} - 1.$$

On one hand, it is easy to check that

$$\frac{1}{n} \sum_{k=1}^n |b_k| = 1.$$

On the other hand, for the subsequences indexed by m_n and h_n respectively, one finds

$$M_{\mathbf{b},1;1}(m_n) = \frac{1}{m_n} \left(\sum_{k=0}^n 2^{2k} - \frac{1}{2} \sum_{k=1}^n 2^{2k} \right) = \frac{1}{3},$$

and

$$M_{\mathbf{b},1;1}(h_n) = \frac{1}{h_n} \left(\sum_{k=0}^n 2^{2k} - \frac{1}{2} \sum_{k=1}^{n+1} 2^{2k} \right) = -\frac{1}{3}.$$

What follows is a simple counterexample for the general failure of Lemma 2.1 if condition (2.1) is not satisfied. Let $\mathbf{b} = (b_k)_{k \in \mathbb{N}}$ and $\mathbf{a} = (a_k)_{k \in \mathbb{N}}$ be defined as follows:

$$\begin{aligned} b_{2n-1} &:= -\sqrt{2n}, & b_{2n} &:= 1 + \sqrt{2n}, & n &\in \mathbb{N}, \\ a_{2n-1} &:= -\frac{1}{\sqrt{2n}}, & a_{2n} &:= \frac{1}{\sqrt{2n}}, & n &\in \mathbb{N}. \end{aligned}$$

Then $a = \lim_n a_n = 0$, and $b = \lim_n M_{\mathbf{b},1;1}(n) = \frac{1}{2}$. Furthermore, as $n \rightarrow +\infty$, first

$$\frac{1}{2n} \sum_{k=1}^{2n} |b_k| = \frac{1}{2} + \frac{1}{n} \sum_{k=1}^n \sqrt{2k} \rightarrow +\infty,$$

and second

$$M_{\mathbf{ab},1;1}(2n) = 1 + \frac{1}{2n} \sum_{k=1}^n \frac{1}{\sqrt{2k}} \rightarrow 1 > 0 = ab.$$

Finally, one can wonder if (3.3) holds true under all the assumptions of Proposition 3.1 but (3.2). The answer is negative as the following example shows for the case $p = 1$, $m = 2$, and $q = m$. Indeed, take

$$b_k = 1, \quad a_{k_1, k_2} = (\sqrt{k_1} - \sqrt{k_1 - 1})\sqrt{k_2}, \quad k, k_1, k_2 \in \mathbb{N}.$$

Then $b = 1$ and $a = \frac{2}{3}$ as

$$\begin{aligned} \lim_n \frac{1}{n^2} \sum_{1 \leq k_1, k_2 \leq n} a_{k_1, k_2} &= \lim_n \frac{1}{n^2} \sum_{1 \leq k_1, k_2 \leq n} (\sqrt{k_1} - \sqrt{k_1 - 1})\sqrt{k_2} \\ &= \lim_n \frac{1}{n^2} \sqrt{n} \sum_{k_2=1}^n \sqrt{k_2} = \lim_n \frac{1}{n} \sum_{k_2=1}^n \sqrt{\frac{k_2}{n}} = \int_0^1 x^{\frac{1}{2}} dx = \frac{2}{3}. \end{aligned}$$

Computing the left hand side of (3.3), we get

$$\begin{aligned} \lim_n \frac{1}{n^3} \sum_{k=1}^n b_k \sum_{k+1 \leq k_1, k_2 \leq n} a_{k_1, k_2} &= \lim_n \frac{1}{n^3} \sum_{k=1}^n (\sqrt{n} - \sqrt{k}) \sum_{k_2=k+1}^n \sqrt{k_2} \\ &= \lim_n \frac{1}{n^2} \sum_{k=1}^n \left(1 - \sqrt{\frac{k}{n}}\right) \sum_{k_2=k+1}^n \sqrt{\frac{k_2}{n}} = \int_0^1 dx (1 - \sqrt{x}) \int_x^1 dy \sqrt{y} \\ &= \frac{4}{15} ab \neq \frac{ab}{3}. \end{aligned}$$

NOTE ADDED IN PROOF

The authors are grateful to O. Kouba who has drawn their attention to Theorem 1 in his note [11] while the present article was in press. The statement of such a theorem is the same as our Theorem 2.2, provided that the involved function f and the sequence $(b_n)_{n \in \mathbb{N}}$ are uniformly continuous on $(0, 1]$ and positive, respectively. By using Weierstrass' Density Theorem as in [11], the former is a corollary of the latter, and can be extended to general p -mean convergent complex-valued sequences $(b_n)_{n \in \mathbb{N}}$, provided that the sequence of their moduli $(|b_n|)_{n \in \mathbb{N}}$ satisfies (2.1).

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REFERENCES

- [1] Accardi L., Crismale V., Lu Y.G. *Constructive universal central limit theorems based on interacting Fock spaces*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **8** (2005), no. 4, 631650.
- [2] Accardi L., Hashimoto Y., Obata N. *Notions of independence related to the free group*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** (1998), 201–220.
- [3] Accardi L., Ji U. C., Saitô K. *The exotic (higher order Lévy) Laplacians generate the Markov processes given by distribution derivatives of white noise*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **16** (2013), 1350020 (26 pages).
- [4] Accardi L., Ji U. C., Saitô K. *Higher order multi-dimensional extensions of Cesàro theorem*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **18** (2015), 1550030 (14 pages).
- [5] Bożejko M., Speicher R. *Completely positive maps on Coxeter groups, deformed commutation relations and operator spaces*, *Math. Ann.* **300** (1994), 97–120.
- [6] Crismale V., Fidaleo F., Lu Y.G. *Ergodic theorems in quantum probability: an application to the monotone stochastic processes*, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), to appear, doi: 10.2422/2036-2145.201506_009, available at arXiv:1505.04688.
- [7] Crismale V., Fidaleo F., Lu Y.G. *From discrete to continuous monotone C^* -algebras via quantum central limit theorems*, preprint (2016).
- [8] Crismale V., Lu Y.G. *Rotation invariant interacting Fock spaces*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** (2007), 211–235.
- [9] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. *Higher transcendental functions Vol. I*, based on notes left by Harry Bateman (Reprint of the 1953 original). Robert E. Krieger Publishing Co., Inc., Fla., Melbourne 1981.
- [10] Feller W. *An introduction to probability theory and its applications. Vol. II*, second edition John Wiley & Sons, Inc., New York-London-Sydney 1971.
- [11] Kouba O. *A generalization of Riemann sums*, in *Mathematical Reflections: two more years (2010-2011)* (ed. Titu Andreescu), XYZ Press (2014), 379–384, available at arXiv:1407.4679.
- [12] Lévy P. *Leçons d'analyse fonctionnelle*, Gauthier-Villars, Paris, 1922.
- [13] Muraki N. *Monotonic independence, monotonic central limit theorem and monotonic law of small numbers*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **4** (2001), 39–58.
- [14] Royden H. L. *Real analysis*, third edition. Macmillan Publishing Company, New York 1988.

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